Received: *October 6, 2000*, Accepted: *October 23, 2000*

Real representation in chiral gauge theories on the lattice

Hiroshi Suzuki

Department of Mathematical Sciences, Ibaraki University
Mito 310-8512, Japan
E-mail: hsuzuki@mito.ipc.ibaraki.ac.jp

Abstract: The Weyl fermion belonging to the real representation of the gauge group provides a simple illustrative example for Lüscher's gauge-invariant lattice formulation of chiral gauge theories. We can explicitly construct the fermion integration measure globally over the gauge-field configuration space in the arbitrary topological sector; there is no global obstruction corresponding to the Witten anomaly. It is shown that this Weyl formulation is equivalent to a lattice formulation based on the Majorana (left-right-symmetric) fermion, in which the fermion partition function is given by the pfaffian with a definite sign, up to physically irrelevant contact terms. This observation suggests a natural relative normalization of the fermion measure in different topological sectors for the Weyl fermion belonging to the complex representation.

Keywords: Renormalization Regularization and Renormalons, Lattice Gauge Field Theories, Gauge Symmetry, Anomalies in Field and String Theories.

JHEP10(2000)039

Contents

1. **Introduction** 1

2. **Real representations in Lüscher's formulation** 2

3. **Global existence of the fermion integration measure** 4

4. **Fermion expectation values** 6

5. **Matching to the Majorana formulation** 9

6. **Relative normalization for different topological sectors** 11

7. **Conclusion** 12

1. Introduction

A general strategy to implement anomaly-free chiral gauge theories on the lattice while preserving the *exact* gauge invariance has emerged recently [1]–[6]. In this paper, we apply this formulation to a single Weyl fermion, which belongs to a *real* representation[1] of the gauge group. Our motivation is two fold:

In the formulation of [1, 3], there are two kinds of obstruction that prevent the gauge-invariant formulation. The first is the *local* gauge anomaly that corresponds to the gauge anomaly in the continuum theory, but requires a control with finite lattice spacings [1]–[7] (see also [8]). The second is the *global* topological obstruction [1, 3, 9], which is a lattice counterpart of the Witten anomaly [10].[2] The local anomaly is absent from real representations, so we expect that global issues in the formulation are highlighted. In fact, we can show that there is no global obstruction for real representations and that it is always possible to construct the gauge-invariant fermion integration measure globally, over the gauge-field configuration space.[3] This is the expected result from the knowledge in the continuum theory [10, 12, 13]. We will

[1]This is sometimes called the real-positive representation in the literature.

[2]The Witten anomaly in lattice gauge theory has been studied also from the viewpoint of the spectral flow [11].

[3]For *pseudo*-real representations of SU(2), from which the local anomaly is also absent, it has been shown [9] that a globally consistent definition of the fermion integration measure is impossible.

explicitly construct such a measure and, with that measure, we can work out all the quantities in the formulation, including fermion expectation values in general topological sectors. In this way, real representations provide an illustrative example for the formulation.

Secondly, there has been a renewed interest [14] in the context of the domain wall fermion [15] on a lattice formulation of SUSY Yang-Mills theories [16]–[18], in which the fermion (the gaugino) belongs to the real representation, i.e. the adjoint representation. Usually such a fermion is regarded as the Majorana fermion because either Weyl or Majorana is a matter of convention in four-dimensional continuum theory and the latter is more symmetric with respect to the chirality. However it is not obvious whether or not the lattice formulation based on the Weyl fermion [1]–[6] and that based on the Majorana fermion are equivalent. We will show that fermion expectation values in general topological sectors differ, in the two formulations, only by contact terms that are irrelevant in physical amplitudes. Thus they are actually physically equivalent. This result supports the view that the framework of [1, 3] provides a unified treatment of chiral gauge theories in general. The matching between the Weyl and the Majorana formulations moreover suggests a natural relative normalization of the fermion integration measure in different topological sectors for the Weyl fermion in the *complex* representation.

We begin with recapitulating some basics of the formulation. For unexplained notations and for more details, see [1, 3]. We assume that the lattice volume is finite throughout this paper.

2. Real representations in Lüscher's formulation

In the formulation of [1, 3], the expectation value of an operator $\mathcal{O}$ in the fermion sector is defined by the path integral

$$\langle\mathcal{O}\rangle_{\rm F}=\int{\rm D}[\psi]{\rm D}[\overline{\psi}]\,\mathcal{O}e^{-S_{\rm F}}. \tag{2.1}$$

In this paper, the fermion action is taken as[4]

$$S_{\rm F}=a^4\sum_x\left[\overline{\psi}(x)D\psi(x)+\frac{1}{2}im\psi^T(x)B\psi(x)-\frac{1}{2}im\overline{\psi}(x)B^{-1}\overline{\psi}^T(x)\right], \tag{2.2}$$

where the Dirac operator D satisfies the Ginsparg-Wilson (GW) relation [19] $\gamma_5 D+D\gamma_5=aD\gamma_5 D$. We require that D be gauge-covariant and that it depends, locally and smoothly, on the gauge field. Such a Dirac operator in fact exists [20, 21]. The locality and the smoothness are, however, guaranteed only in a restricted gauge-field

[4] The matrix B is defined by $B=C\gamma_5$ from the charge conjugation matrix C. We take the representation of the Dirac algebra such that $C\gamma_\mu C^{-1}=-\gamma_\mu^T=-\gamma_\mu^*$, $C\gamma_5C^{-1}=\gamma_5^T=\gamma_5^*$, $C^\dagger C=1$ and $C^T=-C$. These imply $B\gamma_\mu B^{-1}=\gamma_\mu^T=\gamma_\mu^*$, $B^\dagger B=1$ and $B^T=-B$.

configuration space, as expected from the index relation [22]. For the overlap-Dirac operator [21], the sufficient condition is [23]

$$\|1 - R[U(p)]\| < \epsilon \qquad \text{for all plaquettes } p, \tag{2.3}$$

where R is the representation of the gauge group and ϵ is any fixed positive number smaller than 1/30. Under this admissibility, the gauge-field configuration space is divided into topological sectors [24, 1] (see also [25]). As further requirements, we assume the γ_5-hermiticity $D^\dagger = \gamma_5 D \gamma_5$ and the charge conjugation property $D^* = BD_{R\to R^*}B^{-1}$,[5] where R^* is the complex conjugate representation of R. For real representations, we may take $R(T^a)^* = R(T^a) = -R(T^a)^T$, where $R(T^a)$ is the representation matrix for the Lie algebra of the gauge group. This implies $D^* = BDB^{-1}$ for real representations.

In (2.2), we have introduced the "Majorana" mass terms to treat topologically non-trivial sectors, in which there are zero modes of the Dirac operator, just as easily as the vacuum sector. If one is interested in the massless theory, it is sufficient to take the $m \to 0$ limit at the very end of calculations. The mass terms are consistent with the Fermi statistics and, for real representations, gauge-invariant.

The chirality of the Weyl fermion is introduced as follows [26]:[6] The GW chiral matrix is defined by $\hat\gamma_5 = \gamma_5(1 - aD)$.[7] The chiral projectors are then defined by $\hat P_\pm = (1 \pm \hat\gamma_5)/2$ and $P_\pm = (1 \pm \gamma_5)/2$. Since $P_+ D = D\hat P_-$, we can consistently impose the chirality as $\hat P_- \psi = \psi$ and $\overline{\psi} P_+ = \overline{\psi}$. Note that the mass terms in (2.2) are also consistent with this definition of the chirality because $B\hat P_- = \hat P_-^* B = \hat P_-^T B$.

To define the fermion integration measure $\mathrm{D}[\psi]\mathrm{D}[\overline{\psi}]$ in (2.1), one first introduces basis vectors v_j ($j = 1, 2, \dots, \operatorname{Tr}\hat P_-$), which satisfy the constraint $\hat P_- v_j = v_j$ and $(v_j, v_k) = \delta_{jk}$.[8] The fermion field is then expanded as $\psi(x) = \sum_j v_j(x) c_j$ and the measure is defined by $\mathrm{D}[\psi] = \prod_j \mathrm{d}c_j \equiv \mathrm{d}c_1 \mathrm{d}c_2 \cdots \mathrm{d}c_{\operatorname{Tr}\hat P_-}$. These conditions, however, do not specify the measure uniquely; there remains a phase ambiguity that may depend on the gauge field. For a different choice of basis vectors $\tilde v_j$, one has

$$\tilde v_j(x) = \sum_k v_k(x) (\mathcal{Q}^{-1})_{kj}, \tag{2.4}$$

with a unitary matrix $\mathcal{Q}$. The coefficients are thus related as $\tilde c_j = \sum_k \mathcal{Q}_{jk} c_k$ and the measures differ by a phase factor, $\prod_j \mathrm{d}c_j = \det \mathcal{Q} \prod_j \mathrm{d}\tilde c_j$. How to choose (and whether it is possible to choose) the phase over the gauge-field configuration space that is consistent with the gauge invariance is the central issue in the formulation. The measure for the anti-fermion is defined similarly but with respect to P_+ as $\mathrm{D}[\overline{\psi}] = \prod_k \mathrm{d}\bar c_k \equiv$

[5] Throughout this paper, the complex conjugation and the transpose operation on an operator are defined with respect to the corresponding kernel in position space.

[6] For definiteness, we will consider the left-handed Weyl fermion.

[7] Note that $(\hat\gamma_5)^2 = 1$ and $(\hat\gamma_5)^\dagger = \hat\gamma_5$.

[8] The inner product for spinors is defined by $(\psi, \varphi) = a^4 \sum_x \psi^\dagger(x)\varphi(x)$.

$\mathrm{d}\bar{c}_1 \mathrm{d}\bar{c}_2 \cdots \mathrm{d}\bar{c}_{\mathrm{Tr}\, P_+}$, where $\overline{\psi}(x) = \sum_k \bar{c}_k \bar{v}_k(x)$, $\bar{v}_k P_+ = \bar{v}_k$ $(k = 1, 2, \ldots, \mathrm{Tr}\, P_+)$ and $(\bar{v}_k^\dagger, \bar{v}_l^\dagger) = \delta_{kl}$. The phase of $\mathrm{D}[\overline{\psi}]$ can be chosen as being independent of the gauge field and it thus has no physical relevance.

An important point to note is that the above construction refers to a specific topological sector. The number of integration variables in $\mathrm{D}[\psi]$ is $\mathrm{Tr}\, \hat{P}_-$, and this number depends on the gauge-field configuration. In this way, the fermion-number violation in topologically non-trivial sectors is naturally incorporated. Since $\mathrm{Tr}\, \hat{P}_-$ is an integer [22], the smoothness of the Dirac operator in the admissible space (2.3) guarantees that $\mathrm{Tr}\, \hat{P}_-$ is constant within a connected component in the admissible space. The full expectation value, including the gauge field sector, is thus given by

$$\langle \mathcal{O} \rangle = \frac{1}{\mathcal{Z}} \sum_M \int_M \mathrm{D}[U]\, e^{-S_{\mathrm{G}}} \mathcal{N}(M) e^{i\vartheta(M)} \langle \mathcal{O} \rangle_{\mathrm{F}}^M\,, \tag{2.5}$$

where $\mathcal{Z}$ is chosen as $\langle 1 \rangle = 1$ and M stands for each connected component in the admissible space. The restriction of the gauge-field integration to the admissible space may be implemented by the modified plaquette action [1] for example. On the other hand, as already emphasized in [1], at the moment there is no obvious way to fix the relative normalization $\mathcal{N}(M)$ and the relative phase $\vartheta(M)$ for different topological sectors. We will come back to this point in a later section.

3. Global existence of the fermion integration measure

In this section, for real representations, we will show that it is possible to construct the gauge-invariant fermion measure globally and smoothly over the gauge-field configuration space (or more precisely, within each connected component in the admissible space). The underlying symplectic structure plays the key role in this.

Take a certain gauge-field configuration $U(x, \mu)$. We will construct the basis vectors v_j introduced in the previous section starting with a complete set of arbitrarily chosen vectors u_j in the constrained space $\hat{P}_- u_j = u_j$. We first set $v_1 = u_1/\sqrt{(u_1, u_1)}$. Next we can take v_2 as $v_2 = v_1' \equiv B^{-1} v_1^*$, because v_1' satisfies $\hat{P}_- v_1' = v_1'$ and $(v_1, v_1') = 0$, since $B^T = -B$ (v_2 is correctly normalized, $(v_2, v_2) = 1$). Note that $v_2' = B^{-1} v_2^* = -v_1$. Since u_j span a complete set, we have $v_2 = \sum_{j \neq 1} k_j u_j$, where we may assume $k_2 \neq 0$ without loss. Thus, we can replace u_1 and u_2 in the complete set by v_1 and v_2. Next, we define v_3 from u_3 such that it is orthogonal to v_1 and v_2. This can be done by the Gram-Schmidt method as $\tilde{v}_3 = u_3 - (v_1, u_3) v_1 - (v_2, u_3) v_2$ and $v_3 = \tilde{v}_3/\sqrt{(\tilde{v}_3, \tilde{v}_3)}$; v_4 is defined from v_3 by $v_4 = v_3' = B^{-1} v_3^*$. Now we see that v_4 is linearly independent of v_1, v_2 and v_3, because $(v_1, v_4) = -(v_3, v_2) = 0$, $(v_2, v_4) = (v_3, v_1) = 0$ and $(v_3, v_4) = 0$. Since $v_4 = \sum_{j \neq 1,2,3} k_j' u_j$, we may replace u_3 and (say) u_4 in the complete set by v_3 and v_4. Clearly this procedure can be repeated pairwise and we are left with the orthonormal complete set v_j with $\hat{P}_- v_j = v_j$ such that $v_{2l} = v_{2l-1}'$

and $v_{2l-1} = -v'_{2l}$. This basis v_j can be characterized by

$$v'_j(x) = B^{-1} v_j^*(x) = J_{jk} v_k(x) \,, \qquad J_{jk} = \delta_{j+1,k} - \delta_{j,k+1} \,, \tag{3.1}$$

where $J^\dagger J = 1$ and $J^T = -J$.

For a fixed gauge-field configuration $U(x,\mu)$, we have shown that it is always possible to construct v_j such that $\hat{P}_- v_j = v_j$, $(v_j, v_k) = \delta_{jk}$ and (3.1) hold. These v_j moreover can be *smoothly* continued to other gauge-field configurations, at least within a sufficiently small local patch containing $U(x,\mu)$. The reason is that the above construction is purely algebraic and when the gauge field is continuously varied, v_j changes smoothly. The smoothness of the construction breaks down only when, for example, v_2 happens to have no component of u_2 and we need to change the labelling of u_j's. But such a situation cannot occur for sufficiently close neighbors of $U(x,\mu)$.

Therefore, it is always possible to construct a smooth basis v_j within a local patch in the gauge-field configuration space such that (3.1) holds. Now we can show that, as long as condition (3.1) is satisfied, the corresponding measure $\mathrm{D}[\psi]$ — we call this the symplectic measure — is *unique*. The proof of this important fact is simple: assume a different basis $\tilde{v}_j$ also satisfies (3.1). Since v_j and $\tilde{v}_j$ are related by (2.4), (3.1) implies that the unitary matrix $\mathcal{Q}$ satisfies $J\mathcal{Q}J^{-1} = \mathcal{Q}^*$, i.e. $\mathcal{Q}$ is symplectic. Namely, we have $\det \mathcal{Q} = 1$[9] and the associated measure for v_j and that for $\tilde{v}_j$ are identical.

Now, cover the gauge-field configuration space by a collection of local coordinate patches. Within each patch, we can construct the smooth symplectic measure as described above. In an overlap of two patches, the basis vectors in one patch and that in another patch are not necessarily the same. However, corresponding *measures* are identical, since both are symplectic, and the symplectic measure is unique. This shows that it is always possible to define a smooth measure over the gauge-field configuration space. The important point is that the construction of the symplectic measure within a local patch requires only the *local* information, but nevertheless the symplectic condition (3.1) guarantees the *global* consistency of the measure.

Under the infinitesimal variation of the gauge field

$$\delta_\eta U(x,\mu) = a\eta_\mu(x) U(x,\mu) \,, \tag{3.2}$$

the measure changes as $\delta_\eta \mathrm{D}[\psi] = -i\mathcal{L}_\eta \mathrm{D}[\psi]$, where the measure term $\mathcal{L}_\eta$ [1, 3] is defined by $\mathcal{L}_\eta = i\sum_j (v_j, \delta_\eta v_j)$. For the symplectic measure, the measure term identically vanishes, $\mathcal{L}_\eta = i\sum_l [(v_{2l-1}, \delta_\eta v_{2l-1}) + (v_{2l}, \delta_\eta v_{2l})] = i\sum_l \delta_\eta (v_{2l-1}, v_{2l-1}) = 0$, because $v_{2l} = v'_{2l-1} = B^{-1} v^*_{2l-1}$. This implies that the symplectic measure is

[9] We define $\xi' \equiv J^{-1}\xi^*$. If ξ is an eigenvector of $\mathcal{Q}$, $\mathcal{Q}\xi = e^{i\theta}\xi$, then ξ' has the eigenvalue $e^{-i\theta}$. Since ξ and ξ' are linearly independent and $\xi'' = -\xi$, this implies that the eigenvalues of $\mathcal{Q}$ always come in pairs as $e^{i\theta}$ and $e^{-i\theta}$.

independent of the gauge field within a connected component in the admissible space. Incidentally, since the measure term transforms as $\widetilde{\mathcal{L}}_\eta = \mathcal{L}_\eta - i\delta_\eta \ln\det \mathcal{Q}$ under the change of basis vectors (2.4), *any* measure with vanishing measure term $\mathcal{L}_\eta = 0$ is identical to the symplectic measure up to a constant phase.

It remains to be shown that the symplectic measure is gauge-invariant. The infinitesimal gauge transformation is given by $\eta_\mu(x) = -\nabla_\mu \omega(x)$ in (3.2).[10] By using the gauge covariance of the Dirac operator $\delta_\eta D = [R(\omega), D]$ in (2.1), we have as the gauge variation of $\langle \mathcal{O} \rangle_{\mathrm{F}}$,

$$\delta_\eta \langle \mathcal{O} \rangle_{\mathrm{F}} = \langle \delta_\eta \mathcal{O} \rangle_{\mathrm{F}} + [\mathrm{Tr}\, R(\omega)(P_+ - \hat{P}_-) - i\mathcal{L}_\eta] \langle \mathcal{O} \rangle_{\mathrm{F}} \,. \tag{3.3}$$

In the quantity in square brackets, the first term comes from the jacobian of the change of fermion variables and the second term from the fact that basis vectors themselves change under (3.2). We showed that $\mathcal{L}_\eta = 0$ for the symplectic measure. On the other hand, noting $P_+^T = BP_+B^{-1}$, $R(\omega)^T = -R(\omega)$ and $\hat{P}_-^T = B\hat{P}_-B^{-1}$ for real representations, we see that the first term identically vanishes. Namely, expectation values of gauge invariant operators are always gauge-invariant and the symplectic measure (or more generally any measure with $\mathcal{L}_\eta = 0$) is gauge-invariant.[11] This establishes the existence of a globally consistent gauge-invariant measure in any topological sector; there is no global obstruction for real representations.

4. Fermion expectation values

In this section, we explicitly compute the expectation value (2.1) by using the symplectic measure. As shown in the previous section, the symplectic measure can be constructed starting with any complete set u_j satisfying $\hat{P}_- u_j = u_j$. A particularly convenient complete set u_j is provided by eigenvectors of the hermitean operator $D^\dagger D = (\gamma_5 D)^2$:

$$D^\dagger D u_j(x) = \lambda_j^2 u_j(x) \,, \qquad \hat{P}_- u_j(x) = u_j(x) \,. \tag{4.1}$$

(This choice is analogous to that in the treatment of covariant gauge anomalies in the continuum theory [27].) These two conditions are consistent because $D^\dagger D$ and $\hat{P}_-$ commute. For later comparison with the Majorana formulation, we need to know

[10] $\nabla_\mu \omega(x) = [U(x,\mu)\omega(x + a\hat{\mu})U(x,\mu)^{-1} - \omega(x)]/a$ is the covariant difference operator.

[11] For anomaly-free *complex* representations, the quantity $\mathrm{Tr}\, R(\omega)(P_+ - \hat{P}_-)$ does not vanish and the way to (and whether it is possible to) choose $\mathcal{L}_\eta$ to eliminate the combination inside the square brackets is the aforementioned problem of the local gauge anomaly. This problem can be studied by cohomological techniques [7, 3, 4]. The current status of our knowledge concerning $\mathcal{L}_\eta$ is as follows: when the gauge group is U(1), such $\mathcal{L}_\eta$ has been known non-perturbatively on finite lattices [1]. For general compact gauge groups, $\mathcal{L}_\eta$ has been known, but only to all orders in the perturbation theory on the infinite lattice [4, 6]. For the representation in the electroweak $\mathrm{SU}(2) \times \mathrm{U}(1)$, $\mathcal{L}_\eta$ has been known non-perturbatively at least on the infinite lattice [5].

JHEP10(2000)039

some details concerning the eigenvalue problem (4.1). For this, we consider the auxiliary problem[12]

$$\gamma_5 D\varphi_n(x) = \lambda_n \varphi_n(x), \qquad \lambda_n\text{: real}\,, \quad n = 1, 2, \cdots, \mathrm{Tr}\, 1\,. \tag{4.2}$$

The eigenvectors φ_n are classified into three categories:[13] (i) $\lambda_n \neq 0$ and $\lambda_n \neq \pm 2/a$. Then $\widetilde{\varphi}_n \equiv \gamma_5(1 - aD/2)\varphi_n/\sqrt{1 - a^2\lambda_n^2/4}$ has the eigenvalue $-\lambda_n$; the eigenvalues thus come in pairs as λ_n and $-\lambda_n$. (ii) $\lambda_n = \pm 2/a$. Denoting $\Psi_\pm$ as the corresponding eigenvectors, one has $P_\pm \Psi_\pm = \hat{P}_\mp \Psi_\pm = \Psi_\pm$. We denote the number of $\Psi_\pm$ as $N_\pm$. (iii) $\lambda_n = 0$. One can choose the eigenvectors with definite chiralities as $P_\pm \varphi_0^\pm = \hat{P}_\pm \varphi_0^\pm = \varphi_0^\pm$. We denote the number of $\varphi_0^\pm$ as $n_\pm$. The number $n_+ - n_-$ is the analytic index on the lattice [22], which is constant in a connected component in the admissible space. For the number of eigenvectors of the latter two categories, one can show the index relation [28]

$$n_+ - n_- + N_+ - N_- = 0\,, \tag{4.3}$$

starting with $\mathrm{Tr}\,\gamma_5 = 0$. For *real* representations, all the eigenvalues *including* $\lambda_n = 0$ and $\lambda_n = \pm 2/a$ are moreover *doubly-degenerate*: φ_n and $\varphi_n' = B^{-1}\varphi_n^*$ give the same eigenvalue λ_n, and φ_n' is linearly independent with φ_n because $(\varphi_n, \varphi_n') = 0$. In particular, $N_\pm$ and $n_\pm$ are even numbers.

Once having obtained the solution of (4.2), we can obtain all the solutions of (4.1) by simply multiplying $\hat{P}_-$, because φ_n span a complete set. In this way, we have: (I) u_j with $\lambda_j^2 \neq 0$ and $\lambda_j^2 \neq 4/a^2$ from category (i). But since $\hat{P}_-[\widetilde{\varphi}_n + (1 - a\lambda_n/2)\varphi_n] = 0$, only one linear combination of φ_n and $\widetilde{\varphi}_n$ gives rise to the solution of (4.1). Thus the total number of this type of u_j is $[\mathrm{Tr}\,1 - (N_+ + N_- + n_+ + n_-)]/2$. (II) u_j with $\lambda_j^2 = 4/a^2$. This is given by $\hat{P}_-\Psi_+$ and the total number is N_+. (III) u_j with $\lambda_j^2 = 0$. This is given by $\hat{P}_-\varphi_0^-$ and the total number is n_-.

Following the previous construction from u_j to v_j, we thus obtain v_j that satisfy (4.1), (3.1) and $(v_j, v_k) = \delta_{jk}$. Below we will use this particular basis to compute the expectation value (2.1). Recall, however, that the fermion measure itself is independent of which kind of basis vectors are employed, as long as the symplectic condition (3.1) is satisfied.

The expectation value (2.1) also depends on how we choose the phase of $\mathrm{D}[\overline{\psi}]$. We fix this phase by the following natural mapping from v_j to $\overline{v}_j$ for $\lambda_j \neq 0$:

$$\overline{v}_j(x) = \frac{1}{\lambda_j} v_j^\dagger D^\dagger(x)\,, \qquad \lambda_j > 0\,. \tag{4.4}$$

Note that $\overline{v}_j$ so constructed satisfies $\overline{v}_j P_+ = \overline{v}_j$, $\overline{v}_j DD^\dagger = \overline{v}_j(D\gamma_5)^2 = \lambda_j^2 \overline{v}_j$, $v_j = D^\dagger \overline{v}_j^\dagger/\lambda_j$ and $(\overline{v}_j^\dagger, \overline{v}_k^\dagger) = \delta_{jk}$. This mapping gives rise to the symplectic structure

[12] Note that $\gamma_5 D$ is hermitean.

[13] Since it is simple to prove the following statements, we do not give the detailed proof.

$\overline{v}'_j \equiv \overline{v}^*_j B = J_{jk}\overline{v}_k$ also for $\overline{v}_j$. For the zero modes $\lambda_j^2 = 0$, a mismatch between $\overline{v}_j$ and v_j may occur and we can take $\varphi_0^{+\dagger} P_+ = \varphi_0^{+\dagger}$ as the basis vectors for the zero modes in $\overline{v}_j$. The total number of these is n_+.

We have completely fixed the phase ambiguity for the measure in (2.1). What remains to be done is simply the Grassmann integrals with respect to c_j and $\overline{c}_k$. For the partition function $\langle 1 \rangle_{\rm F}$, after a careful calculation using the above relations, we have[14]

$$\langle 1 \rangle_{\rm F} = \prod_{\substack{\lambda_n > 0 \\ \lambda_n \neq 2/a}} \left[-(\lambda_n^2 + m^2) \right] \left[-\left(\frac{4}{a^2} + m^2 \right) \right]^{N_+/2} (im)^{(n_+ + n_-)/2} \tag{4.5}$$

$$= i^{[{\rm Tr}\, 1 - (n_+ - n_-)]/2} \prod_{\substack{\lambda_n > 0 \\ \lambda_n \neq 2/a}} (\lambda_n^2 + m^2) \left(\frac{4}{a^2} + m^2 \right)^{N_+/2} m^{(n_+ + n_-)/2}, \tag{4.6}$$

in terms of the eigenvalues λ_n in (4.2). In this expression, the product $\prod_{\substack{\lambda_n > 0 \\ \lambda_n \neq 2/a}}$ is understood to be taken *without* counting the double degeneracy of λ_n (i.e. one factor for each λ_n). We have used the index relation (4.3) in deriving the second line.

The expression (4.6) holds for any topological sector. Interestingly, in the massive theory, the partition function $\langle 1 \rangle_{\rm F}$ has a definite sign, up to a proportionality constant that depends only on which topological sector is concerned through the combination $n_+ - n_-$.[15] In the massless theory $m \to 0$, $\langle 1 \rangle_{\rm F}$ vanishes when there exists a zero mode, as should be the case. The general fermion expectation value $\langle \mathcal{O} \rangle_{\rm F}$ is computed as usual by $\langle 1 \rangle_{\rm F}$ times the Wick contractions of fermion fields. The basic contractions are given by

$$\begin{aligned}
\frac{\langle \psi(x) \overline{\psi}(y) \rangle_{\rm F}}{\langle 1 \rangle_{\rm F}} &= \hat{P}_- \frac{1}{D^\dagger D + m^2} D^\dagger P_+(x, y) \\
\frac{\langle \psi(x) \psi^T(y) \rangle_{\rm F}}{\langle 1 \rangle_{\rm F}} &= \hat{P}_- \frac{-im}{D^\dagger D + m^2} B^{-1} \hat{P}_-^T(x, y) \\
\frac{\langle \overline{\psi}^T(x) \overline{\psi}(y) \rangle_{\rm F}}{\langle 1 \rangle_{\rm F}} &= P_+^T B \frac{im}{D D^\dagger + m^2} P_+(x, y) \,.
\end{aligned} \tag{4.7}$$

It is easy to express these basic contractions in terms of the eigenvalues and eigenfunctions in (4.2) by noting $\hat{P}_-(x, y) = \sum_j v_j(x) v_j^\dagger(y)$ etc., although we do not write them down explicitly. For example, in the massless limit $m \to 0$, $\langle \psi(x) \psi^T(y) \rangle_F / \langle 1 \rangle_F \to -i\varphi_0^-(x) \varphi_0^{-\dagger}(y) B^{-1}/m$ and it thus precisely cancels one m in (4.6) due to one pair

[14]In the massless limit $m \to 0$, this expression may be interpreted as $\langle 1 \rangle_{\rm F} = \pm\sqrt{\det \gamma_5 D}$, as naively expected for the Weyl fermion in a real representation. Since eigenvalues of $\gamma_5 D$ are doubly-degenerate, even if some of the eigenvalues cross zero according to a deformation of the gauge field, there is no ambiguity in the sign of the square root [10] because it is always an even number of eigenvalues that cross zero [13]. This explains (for the vacuum sector) why the Witten anomaly does not appear for real representations from the viewpoint of the spectral flow.

[15]This fact may be of interest from the viewpoint of numerical simulations.

of left-handed zero modes. In this way, any fermion expectation value in any topological sector is obtained by combining (4.6) and (4.7). Note that, according to the above expressions, expectation values of gauge-invariant operators are manifestly gauge-invariant.

5. Matching to the Majorana formulation

As noted in the introduction, in four-dimensional continuum (unregularized) theories, the Weyl fermion in the real representation is equivalent to the Majorana fermion. Thus it is of interest to see how this equivalence is realized in the present formulation in which the left-right chiralities are treated asymmetrically. The lattice implementation of the Majorana (left-right symmetric) fermion would be given by

$$S_{\rm F}^{\rm Majorana} = a^4 \sum_x \left[\frac{1}{2}\chi^T(x) C D \chi(x) + \frac{1}{2} i m \chi^T(x) C\gamma_5 \chi(x) \right], \tag{5.1}$$

where χ is a four-component *unconstrained* spinor field. Note that $(CD)^T = -CD$ and $(C\gamma_5)^T = -C\gamma_5$ being consistent with the Fermi statistics and that the mass term is gauge-invariant for real representations. The expectation value is then given by $\langle \mathcal{O} \rangle_{\rm F}^{\rm Majorana} = \int {\rm D}[\chi]\, \mathcal{O} e^{-S_{\rm F}^{\rm Majorana}}$, where the fermion integration measure is defined by $\chi(x) = \sum_n \varphi_n(x) b_n$ (φ_n's are certain orthonormal basis vectors) and ${\rm D}[\chi] = \prod_n {\rm d}b_n \equiv {\rm d}b_1 {\rm d}b_2 \cdots {\rm d}b_{\rm Tr\,1}$. The important difference from the Weyl formulation is that the Majorana formulation can be set up without referring to a particular topological sector, because the number of integration variables is always the same. Namely, the above definition is uniform for all topological sectors.[16] This property of the Majorana formulation has an interesting implication, as we will discuss in the next section.

We can take the eigenvectors in (4.2) as the basis vectors φ_n. With this choice, we obtain as the fermion partition function

$$\begin{aligned} \langle 1 \rangle_{\rm F}^{\rm Majorana} &= \prod_{\substack{\lambda_n > 0 \\ \lambda_n \neq 2/a}} (\lambda_n^2 + m^2) \left(-\frac{2}{a} - im \right)^{N_+/2} \left(\frac{2}{a} - im \right)^{N_-/2} (-im)^{(n_+ + n_-)/2} \\ &= -i^{-(n_+ - n_-)/2} \left(\frac{2}{a} - im \right)^{(n_+ - n_-)/2} \prod_{\substack{\lambda_n > 0 \\ \lambda_n \neq 2/a}} (\lambda_n^2 + m^2) \left(\frac{4}{a^2} + m^2 \right)^{N_+/2} m^{(n_+ + n_-)/2}, \end{aligned} \tag{5.2}$$

where from the first line to the second line we have used (4.3) and the fact that n_- is an even number. Note that $N_\pm$ appear symmetrically in the first expression, because of the left-right-symmetric treatment in the Majorana formulation.

From (5.1), the fermion partition function in the Majorana formulation is given by the pfaffian $\langle 1 \rangle_{\rm F}^{\rm Majorana} \propto {\rm Pf}(CD + imC\gamma_5)$ and (5.2) gives the precise meaning of

[16] This is analogous to the situation for the Dirac fermion in lattice QCD in which one usually never worries about the relative weight for the fermion measure in different topological sectors.

this pfaffian. In the massless limit, $\langle 1\rangle_{\rm F}^{\rm Majorana}\propto {\rm Pf}\, CD$ and, when the overlap-Dirac operator [21] is employed as D, this coincides with the expression in [18], which is based on a factorization property of the domain wall [15, 14] (with the infinite five-dimensional separation) or the overlap [29] fermion determinant in vector-like theories. In this limit, (5.2) reduces to $\langle 1\rangle_{\rm F}^{\rm Majorana}=-\prod_{\substack{\lambda_n>0\\ \lambda_n\neq 2/a}}\lambda_n^2(4/a^2)^{N_+/2}$ (assuming there is no zero mode), which manifestly has a definite sign. This is important from the viewpoint of numerical simulations [13, 14, 16, 17, 18], because the fermion partition function then allows a statistical weight interpretation. This property with the overlap-Dirac operator has been shown [18] by appealing to the limiting procedure from the domain wall fermion with finite five-dimensional separation. Here we have shown the same property by using general properties of the GW Dirac operator alone.

Comparing (5.2) and (4.6), we find

$$\langle 1\rangle_{\rm F}^{\rm Majorana}=-(-1)^{{\rm Tr}\,1/4}\left(\frac{2}{a}-im\right)^{(n_+-n_-)/2}\langle 1\rangle_{\rm F}^{\rm Weyl}\,. \tag{5.3}$$

Namely, two formulations match up to a proportionality constant that depends only on the topological sector. If one is concerned with a particular topological sector, two formulations are therefore completely equivalent. For the basic contraction, we find

$$\begin{aligned}\frac{\langle\chi(x)\chi^T(y)\rangle_{\rm F}^{\rm Majorana}}{\langle 1\rangle_{\rm F}^{\rm Majonara}}&=\frac{1}{(\gamma_5D)^2+m^2}(D^\dagger-im\gamma_5)C^{-1}(x,y)\\&=\frac{2/a}{2/a-im}\frac{\left\langle\left[\psi(x)-C^{-1}\overline{\psi}^T(x)\right]\left[\psi^T(y)-\overline{\psi}(y)C^{-1T}\right]\right\rangle_{\rm F}^{\rm Weyl}}{\langle 1\rangle_{\rm F}^{\rm Weyl}}-\\&\quad-\frac{1}{2/a-im}\gamma_5C^{-1}a^{-4}\delta_{x,y}\,,\end{aligned}\tag{5.4}$$

where, in deriving the last expression, we have noted $1=P_++\hat{P}_--a\gamma_5D/2$. The relation in the opposite direction is given by

$$\begin{aligned}\frac{\langle\psi(x)\overline{\psi}(y)\rangle_{\rm F}^{\rm Weyl}}{\langle 1\rangle_{\rm F}^{\rm Weyl}}&=-\frac{2/a}{2/a-im}\hat{P}_-\frac{\langle\chi(x)\chi^T(y)\rangle_{\rm F}^{\rm Majorana}}{\langle 1\rangle_{\rm F}^{\rm Majonara}}C^TP_+\\\frac{\langle\psi(x)\psi^T(y)\rangle_{\rm F}^{\rm Weyl}}{\langle 1\rangle_{\rm F}^{\rm Weyl}}&=\frac{2/a}{2/a-im}\left[\hat{P}_-\frac{\langle\chi(x)\chi^T(y)\rangle_{\rm F}^{\rm Majorana}}{\langle 1\rangle_{\rm F}^{\rm Majonara}}\hat{P}_-^T-\frac{a}{2}\hat{P}_-(x,y)\gamma_5C^{-1}\right]\\\frac{\langle\overline{\psi}^T(x)\overline{\psi}(y)\rangle_{\rm F}^{\rm Weyl}}{\langle 1\rangle_{\rm F}^{\rm Weyl}}&=\frac{2/a}{2/a-im}\left[P_+^TC\frac{\langle\chi(x)\chi^T(y)\rangle_{\rm F}^{\rm Majorana}}{\langle 1\rangle_{\rm F}^{\rm Majonara}}C^TP_++\frac{a}{2}CP_+a^{-4}\delta_{x,y}\right].\end{aligned}\tag{5.5}$$

Therefore, with these rules (5.4) and (5.5), the expectation values are identical in the two formulations, up to contact terms.[17] In particular, they lead to the same physical amplitudes with that matching rule.

[17]Since the kernel $\hat{P}_-(x,y)$ decays exponentially with a fixed range in the lattice units [23], this can effectively be regarded as a contact term in the continuum limit.

6. Relative normalization for different topological sectors

We have seen that there is a complete matching between the Weyl formulation and the Majorana formulation for real representations. In this section, we present a possible implication of this matching for the relative normalization of the fermion measure in different topological sectors (the factor $\mathcal{N}(M)$ in (2.5)) for the Weyl fermion belonging to the *complex* representation. For complex representations, the mass term breaks the gauge symmetry. We thus restrict our problem to the massless theory.

Suppose that we have a consistent gauge-invariant measure for the Weyl fermion belonging to the *complex* representation, which is specified by the basis vectors v_j. Then the set of vectors $B^{-1}v_j^*$ naturally provides a consistent gauge-invariant measure for the complex conjugate representation R^*. With this choice of measure for R^*, we have $\langle\mathcal{O}\rangle_{\mathrm{F},R}^* = \langle\mathcal{O}^*\rangle_{\mathrm{F},R^*}$ and, as naively expected [12],

$$
\begin{aligned}
\left|\langle\mathcal{O}\rangle_{\mathrm{F},R}\right|^2 &= \langle\mathcal{O}^*\rangle_{\mathrm{F},R^*}\langle\mathcal{O}\rangle_{\mathrm{F},R} \\
&= \langle\mathcal{O}^*\mathcal{O}\rangle_{\mathrm{F},R\oplus R^*}\,,
\end{aligned} \tag{6.1}
$$

where the measure for the *real* representation $R\oplus R^*$ is specified by the basis vectors $V_{2l-1} = (v_l, 0)^T$ and $V_{2l} = (0, B^{-1}v_l^*)^T$. This measure is symplectic with respect to $V_j' \equiv \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)B^{-1}V_j^*$ and we can thus apply the previous arguments. In particular, (6.1) shows that we can compute the modulus of $\langle\mathcal{O}\rangle_{\mathrm{F},R}$ by using (4.6) and (4.7) with $m\to 0$.

For the real representation $R\oplus R^*$, we may use also the Majorana formulation. From (6.1), (5.3) and (5.5), we know that for a fixed topological sector:[18]

$$
\left|\langle\mathcal{O}\rangle_{\mathrm{F},R}\right|^2 = -(-1)^{\mathrm{Tr}\,1/4}\left(\frac{a}{2}\right)^{n_+-n_-}\langle\mathcal{O}^*\mathcal{O}\rangle_{\mathrm{F},R\oplus R^*}^{\mathrm{Majorana}}\,, \tag{6.2}
$$

up to contact terms.[19] In this expression, $n_\pm$ refer to the numbers of zero modes of the *original* Weyl fermion in the complex representation R. Now, as already emphasized, the Majorana formulation is *uniform* for all topological sectors. Thus it is quite natural to adjust the normalization of $\langle\mathcal{O}\rangle_{\mathrm{F,R}}$ as it coincides with the normalization of the Majorana formulation for all topological sectors. Namely, we may define the relative weight for a topological sector as

$$
\langle\mathcal{O}\rangle_{\mathrm{F}} \to \left(\frac{2}{a}\right)^{(n_+-n_-)/2} e^{i\vartheta}\langle\mathcal{O}\rangle_{\mathrm{F}}\,. \tag{6.3}
$$

---

[18] Although the *phase* of the proportionality constant in this expression depends on a way we specified the phase of $\mathrm{D}[\chi]$, this does not affect the following argument for the *normalization* $\mathcal{N}(M)$.

[19] Eq. (5.5) shows that the substitution rule from the Weyl formulation to the Majorana formulation is given by $\psi(x)\to\hat{P}_-\chi(x)$ and $\overline{\psi}(x)\to -\chi^T(x)C^TP_+$.

JHEP10(2000)039

The natural prescription for the full expectation value would thus be

$$\langle\mathcal{O}\rangle = \frac{1}{\mathcal{Z}}\sum_M \int_M \mathrm{D}[U]\, e^{-S_{\mathrm{G}}} \left(\frac{2}{a}\right)^{(n_+-n_-)/2} e^{i\vartheta(M)} \langle\mathcal{O}\rangle_{\mathrm{F}}^M\,, \tag{6.4}$$

where the relative phase $\vartheta(M)$ cannot be fixed from the present argument. Assuming that the operator $\mathcal{O}$ has a definite mass dimension, the dimensionful factor $(1/a)^{(n_+-n_-)/2}$ compensates changes of the mass dimension of $\langle\mathcal{O}\rangle_{\mathrm{F}}^M$, which depends on $\mathrm{Tr}(P_+ - \hat{P}_-) = n_+ - n_-$ (note that the mass dimension of the Grassmann integration $\mathrm{d}c_j$ is $1/2$). This is a natural requirement for $\mathcal{N}(M)$. On the other hand, the relative normalization $2^{(n_+-n_-)/2}$ was determined from the matching with the Majorana formulation. If one chooses the normalization of the GW relation as $D\gamma_5 + \gamma_5 D = kaD\gamma_5 D$, the number 2 changes to $2/k$. Therefore, the Weyl formulation will automatically give rise to the natural relative normalization by choosing the normalization of the Dirac operator as $k = 2$.

7. Conclusion

The real representation, owing to its simplicity with regard to the local gauge anomaly, provides an interesting example with which one can work out all the quantities in Lüscher's gauge-invariant lattice formulation. We hope that we clearly illustrated some global issues in the formulation with this simple example. An interesting implication of the present analysis is that the matching to the Majorana formulation provides a natural normalization of the fermion-integration measure in different topological sectors. This could be physically relevant, for example, when considering the absolute magnitude of fermion-number-violating processes in chiral gauge theories.

The question raised by Taku Izubuchi many years ago initially motivated the present work. I am indebted to Yoshio Kikukawa for enjoyable discussions and to Martin Lüscher for helpful discussions and suggestions, which quite enriched the contents of this paper. I am grateful to members of the CERN Theory Division, where this work was done, especially Patricia Ball, Pilar Hernández, Karl Jansen and Hartmut Wittig for their kind help.

References

[1] M. Lüscher, *Abelian chiral gauge theories on the lattice with exact gauge invariance*, *Nucl. Phys.* **B 549** (1999) 295 [hep-lat/9811032].

[2] H. Suzuki, *Gauge invariant effective action in abelian chiral gauge theory on the lattice*, *Prog. Theor. Phys.* **101** (1999) 1147 [hep-lat/9901012].

[3] M. Lüscher, *Weyl fermions on the lattice and the nonabelian gauge anomaly*, *Nucl. Phys.* **B 568** (2000) 162 [hep-lat/9904009]; *Chiral gauge theories on the lattice with exact gauge invariance*, *Nucl. Phys.* **83** *(Proc. Suppl.)* (2000) 34 [hep-lat/9909150].

[4] H. Suzuki, *Anomaly cancellation condition in lattice gauge theory*, *Nucl. Phys.* **B 585** (2000) 471 [hep-lat/0002009].

[5] Y. Kikukawa and Y. Nakayama, *Gauge anomaly cancellations in* $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ *electroweak theory on the lattice*, hep-lat/0005015.

[6] M. Lüscher, *Lattice regularization of chiral gauge theories to all orders of perturbation theory*, *J. High Energy Phys.* **06** (2000) 028 [hep-lat/0006014].

[7] M. Lüscher, *Topology and the axial anomaly in abelian lattice gauge theories*, *Nucl. Phys.* **B 538** (1999) 515 [hep-lat/9808021];
T. Fujiwara, H. Suzuki and K. Wu, *Noncommutative differential calculus and the axial anomaly in abelian lattice gauge theories*, *Nucl. Phys.* **B 569** (2000) 643 [hep-lat/9906015]; *Axial anomaly in lattice abelian gauge theory in arbitrary dimensions*, *Phys. Lett.* **B 463** (1999) 63 [hep-lat/9906016]; *Application of noncommutative differential geometry on lattice to anomaly analysis in abelian lattice gauge theory*, hep-lat/9910030.

[8] H. Neuberger, *Geometrical aspects of chiral anomalies in the overlap*, *Phys. Rev.* **D 59** (1999) 085006 [hep-lat/9802033];
D.H. Adams, *A topological aspect of the non-abelian anomaly for Weyl fermions on the lattice*, hep-lat/9910036 to appear on *Phys. Rev. Lett.*; *Global obstructions to gauge invariance in chiral gauge theory on the lattice*, *Nucl. Phys.* **B 589** (2000) 633 [hep-lat/0004015].

[9] O. Bär and I. Campos, *Global anomalies in chiral lattice gauge theory*, *Nucl. Phys.* **83** *(Proc. Suppl.)* (2000) 594 [hep-lat/9909081]; *Global anomalies in chiral gauge theories on the lattice*, *Nucl. Phys.* **B 581** (2000) 499 [hep-lat/0001025].

[10] E. Witten, *An* SU(2) *anomaly*, *Phys. Lett.* **B 117** (1982) 324;
S. Elitzur and V.P. Nair, *Nonperturbative anomalies in higher dimensions*, *Nucl. Phys.* **B 243** (1984) 205.

[11] H. Neuberger, *Witten's* SU(2) *anomaly on the lattice*, *Phys. Lett.* **B 437** (1998) 117 [hep-lat/9805027].

[12] L. Alvarez-Gaumé and E. Witten, *Gravitational anomalies*, *Nucl. Phys.* **B 234** (1984) 269.

[13] S.D. H. Hsu, *Gaugino determinant in supersymmetric Yang-Mills theory*, *Mod. Phys. Lett.* **A 13** (1998) 673 [hep-th/9704149].

[14] D.B. Kaplan and M. Schmaltz, *Supersymmetric Yang-Mills theories from domain wall fermions*, *Chin. J. Phys.* **38** (2000) 543 [hep-lat/0002030];
G.T. Fleming, J.B. Kogut and P.M. Vranas, *Super Yang-Mills on the lattice with domain wall fermions*, hep-lat/0008009.

JHEP10(2000)039

[15] D.B. Kaplan, *A method for simulating chiral fermions on the lattice*, *Phys. Lett.* **B 288** (1992) 342 [hep-lat/9206013]; *Chiral fermions on the lattice*, *Nucl. Phys.* **30** *(Proc. Suppl.)* (1993) 597;
Y. Shamir, *Chiral fermions from lattice boundaries*, *Nucl. Phys.* **B 406** (1993) 90 [hep-lat/9303005]; *Lattice chiral fermions*, *Nucl. Phys.* **47** *(Proc. Suppl.)* (1996) 212 [hep-lat/9509023];
V. Furman and Y. Shamir, *Axial symmetries in lattice QCD with Kaplan fermions*, *Nucl. Phys.* **B 439** (1995) 54 [hep-lat/9405004].

[16] G. Curci and G. Veneziano, *Supersymmetry and the lattice: a reconciliation?*, *Nucl. Phys.* **B 292** (1987) 555;
I. Montvay, *An algorithm for gluinos on the lattice*, *Nucl. Phys.* **B 466** (1996) 259 [hep-lat/9510042]; *Supersymmetric gauge theories on the lattice*, *Nucl. Phys.* **53** *(Proc. Suppl.)* (1997) 853 [hep-lat/9607035]; *SUSY on the lattice*, *Nucl. Phys.* **63** *(Proc. Suppl.)* (1998) 108 [hep-lat/9709080]; *Simulation of QCD and other similar theories*, *Nucl. Phys.* **83** *(Proc. Suppl.)* (2000) 188 [hep-lat/9909020];
G. Koutsoumbas and I. Montvay, *Gluinos on the lattice: quenched calculations*, *Phys. Lett.* **B 398** (1997) 130 [hep-lat/9612003];
G. Koutsoumbas, I. Montvay, A. Pap, K. Spanderen, T. Talkenberger, J. Westphalen, *Numerical study of* SU(2) *Yang-Mills theory with gluinos*, *Nucl. Phys.* **63** *(Proc. Suppl.)* (1998) 727 [hep-lat/9709091];
DESY-Munster collaboration, R. Kirchner, S. Luckmann, I. Montvay, K. Spanderen and J. Westphalen, *Numerical simulation of dynamical gluinos: experience with a multi-bosonic algorithm and first results*, *Nucl. Phys.* **73** *(Proc. Suppl.)* (1999) 828 [hep-lat/9808024]; *Evidence for discrete chiral symmetry breaking in $N = 1$ supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 446** (1999) 209 [hep-lat/9810062];
DESY-Munster collaboration, I. Campos et al., *Monte Carlo simulation of* SU(2) *Yang-Mills theory with light gluinos*, *Eur. Phys. J.* **C 11** (1999) 507 [hep-lat/9903014];
DESY-Munster collaboration, A. Fao et al., *Numerical simulations of dynamical gluinos in* SU(3) *Yang-Mills theory: first results*, *Nucl. Phys.* **83** *(Proc. Suppl.)* (2000) 661 [hep-lat/9909070]; *Low-energy features of* SU(2) *Yang-Mills theory with light gluinos*, *Nucl. Phys.* **83** *(Proc. Suppl.)* (2000) 670 [hep-lat/9909071];
A. Donini and M. Guagnelli, *Hybrid molecular dynamics for lattice supersymmetry*, *Phys. Lett.* **B 383** (1996) 301 [hep-lat/9605010];
A. Donini, M. Guagnelli, P. Hernández and A. Vladikas, *Quenched spectroscopy for the $N = 1$ super Yang-Mills theory*, *Nucl. Phys.* **63** *(Proc. Suppl.)* (1998) 718 [hep-lat/9708006]; *Towards $N = 1$ super Yang-Mills on the lattice*, *Nucl. Phys.* **B 523** (1998) 529 [hep-lat/9710065];
A. Donini, E. Gabrielli and M.B. Gavela, *Quenched supersymmetry*, *Nucl. Phys.* **73** *(Proc. Suppl.)* (1999) 721 [hep-lat/9809111]; *Quenched supersymmetry*, *Nucl. Phys.* **B 546** (1999) 119 [hep-th/9810127].

[17] P. Huet, R. Narayanan and H. Neuberger, *Overlap formulation of Majorana-Weyl*

fermions, *Phys. Lett.* **B 380** (1996) 291 [hep-th/9602176];
S. Aoki, K. Nagai and S.V. Zenkin, *Domain wall fermions with Majorana couplings*, *Nucl. Phys.* **B 508** (1997) 715 [hep-lat/9705001]; *Lattice fermions with Majorana couplings*, *Nucl. Phys.* **63** *(Proc. Suppl.)* (1998) 602 [hep-lat/9709058];
J. Nishimura, *Four-dimensional $N = 1$ supersymmetric Yang-Mills theory on the lattice without fine-tuning*, *Phys. Lett.* **B 406** (1997) 215 [hep-lat/9701013]; *Applications of the overlap formalism to super Yang-Mills theories*, *Nucl. Phys.* **63** *(Proc. Suppl.)* (1998) 721 [hep-lat/9709112];
N. Maru and J. Nishimura, *Lattice formulation of supersymmetric Yang-Mills theories without fine-tuning*, *Int. J. Mod. Phys.* **A 13** (1998) 2841 [hep-th/9705152];
T. Hotta, T. Izubuchi and J. Nishimura, *Massless Majorana fermion on the domain wall*, *Nucl. Phys.* **63** *(Proc. Suppl.)* (1998) 685 [hep-lat/9709075]; *Single massless Majorana fermion in the domain-wall formalism*, *Mod. Phys. Lett.* **A 13** (1998) 1667 [hep-lat/9712009];
T. Aoyama and Y. Kikukawa, *Overlap formula for the chiral multiplet*, *Phys. Rev.* **D 59** (1999) 054507 [hep-lat/9803016];
R.G. Edwards, U.M. Heller and R. Narayanan, *Evidence for fractional topological charge in* SU(2) *pure Yang-Mills theory*, *Phys. Lett.* **B 438** (1998) 96 [hep-lat/9806011]; *Evidence for fractional topological charge in* SU(2) *pure Yang-Mills theory*, *Nucl. Phys.* **73** *(Proc. Suppl.)* (1999) 49 7 [hep-lat/9810003]; *Chin. J. Phys.* 38 (2000) 594;
W. Bietenholz, *Exact supersymmetry on the lattice*, *Mod. Phys. Lett.* **A 14** (1999) 51 [hep-lat/9807010];
H. So and N. Ukita, *Ginsparg-Wilson relation and lattice supersymmetry*, *Phys. Lett.* **B 457** (1999) 314 [hep-lat/9812002].

[18] H. Neuberger, *Vector like gauge theories with almost massless fermions on the lattice*, *Phys. Rev.* **D 57** (1998) 5417 [hep-lat/9710089].

[19] P.H. Ginsparg and K.G. Wilson, *A remnant of chiral symmetry on the lattice*, *Phys. Rev.* **D 25** (1982) 2649.

[20] P. Hasenfratz, *Prospects for perfect actions*, *Nucl. Phys.* **63** *(Proc. Suppl.)* (1998) 53 [hep-lat/9709110];
Lattice QCD without tuning, mixing and current renormalization, *Nucl. Phys.* **B 525** (1998) 401 [hep-lat/9802007].

[21] H. Neuberger, *Exactly massless quarks on the lattice*, *Phys. Lett.* **B 417** (1998) 141 [hep-lat/9707022]; *More about exactly massless quarks on the lattice*, *Phys. Lett.* **B 427** (1998) 353 [hep-lat/9801031].

[22] P. Hasenfratz, V. Laliena and F. Niedermayer, *The index theorem in QCD with a finite cut-off*, *Phys. Lett.* **B 427** (1998) 125 [hep-lat/9801021];
M. Lüscher, *Exact chiral symmetry on the lattice and the Ginsparg-Wilson relation*, *Phys. Lett.* **B 428** (1998) 342 [hep-lat/9802011].

JHEP10(2000)039

[23] P. Hernández, K. Jansen and M. Lüscher, *Locality properties of Neuberger's lattice Dirac operator*, *Nucl. Phys.* **B 552** (1999) 363 [hep-lat/9808010].

[24] M. Lüscher, *Topology of lattice gauge fields*, *Comm. Math. Phys.* **85** (1982) 39;
A. Phillips and D. Stone, *Lattice gauge fields, principal bundles and the calculation of topological charge*, *Comm. Math. Phys.* **103** (1986) 599;
A.V. Phillips and D.A. Stone, *The computation of characteristic classes of lattice gauge fields*, *Comm. Math. Phys.* **131** (1990) 255.

[25] T. Fujiwara, H. Suzuki and K. Wu, *Topological charge of lattice abelian gauge theory*, hep-lat/0001029.

[26] F. Niedermayer, *Exact chiral symmetry, topological charge and related topics*, *Nucl. Phys.* **73** *(Proc. Suppl.)* (1999) 105 [hep-lat/9810026];
R. Narayanan, *Ginsparg-Wilson relation and the overlap formula*, *Phys. Rev.* **D 58** (1998) 097501 [hep-lat/9802018];
Y. Kikukawa and A. Yamada, *Axial vector current of exact chiral symmetry on the lattice*, *Nucl. Phys.* **B 547** (1999) 413 [hep-lat/9808026].

[27] K. Fujikawa, *On the evaluation of chiral anomaly in gauge theories with γ_5 couplings*, *Phys. Rev.* **D 29** (1984) 285.

[28] T.-W. Chiu, *The spectrum and topological charge of exactly massless fermions on the lattice*, *Phys. Rev.* **D 58** (1998) 074511 [hep-lat/9804016];
K. Fujikawa, *Relation* $\mathrm{tr}(\gamma_5) = 0$ *and the index theorem in lattice gauge theory*, *Phys. Rev.* **D 60** (1999) 074505 [hep-lat/9904007].

[29] R. Narayanan and H. Neuberger, *Infinitely many regulator fields for chiral fermions*, *Phys. Lett.* **B 302** (1993) 62 [hep-lat/9212019]; *Chiral fermions on the lattice*, *Phys. Rev. Lett.* **71** (1993) 3251 [hep-lat/9308011]; *Chiral determinant as an overlap of two vacua*, *Nucl. Phys.* **B 412** (1994) 574 [hep-lat/9307006]; *A construction of lattice chiral gauge theories*, *Nucl. Phys.* **B 443** (1995) 305 [hep-th/9411108];
R. Narayanan, *Recent developments in chiral gauge theories: approach of infinitely many Fermi fields*, *Nucl. Phys.* **34** *(Proc. Suppl.)* (1994) 95 [hep-lat/9311014];
H. Neuberger, *Chiral fermions on the lattice*, *Nucl. Phys.* **83** *(Proc. Suppl.)* (2000) 67 [hep-lat/9909042];
S. Randjbar-Daemi and J. Strathdee, *On the overlap formulation of chiral gauge theory*, *Phys. Lett.* **B 348** (1995) 543 [hep-th/9412165]; *Chiral fermions on the lattice*, *Nucl. Phys.* **B 443** (1995) 386 [hep-lat/9501027]; *On the overlap prescription for lattice regularization of chiral fermions*, *Nucl. Phys.* **B 466** (1996) 335 [hep-th/9512112]; *Consistent and covariant anomalies in the overlap formulation of chiral gauge theories*, *Phys. Lett.* **B 402** (1997) 134 [hep-th/9703092].